STABLE EQUILIBRIUM OF A DIFFUSION EQUATION ON CONVEX DOMAINS INDUCED BY THE DYNAMICS ON THE BOUNDARY.

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We address the question of existence of nonconstant stable stationary solution (pattern, for short) to the problem

$$\begin{cases} u_t = \triangle u, & (t,x) \in \mathbb{R}^+ \times \Omega \\ u(0,x) = u_0(x) & x \in \Omega \\ \partial_\nu u = \lambda\, f(u), & (t,x) \in \mathbb{R}^+ \times \partial\Omega \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth convex domain, $\lambda \in \mathbb{R}^+$ and f a smooth assigned function.

In [1] the authors, in a computer-assisted work and using bifurcation techniques, give strong evidence that when $f(u) = u - u^3$, $\lambda > 2,84083164$ and Ω the unit square (whence a convex planar domain), then (1) has a pattern.

Such solutions were known to exist for dumbbell type domains [2] and not to exist when Ω is the N-dimensional ball [3].

One way the sphere $\partial B_R(0)$ differs from any other convex hyper-surface $\partial\Omega$ is that the mean curvature of the former is constant whereas it varies in latter case. In order to explore this fact we utilize convenient coordinates to write

$$\triangle v = \triangle_{\mathcal{M}} v + (N-1)\, H(\cdot)\, \partial_\nu v + \partial_\nu^2 v \quad \text{on} \quad \mathcal{M} = \partial\Omega, \tag{2}$$

where $H(\cdot)$ is the mean curvature of $\mathcal{M}$ and $\triangle_{\mathcal{M}} v = g^{\alpha\beta} u_{,\alpha\beta}$ is the Laplace-Beltrami operator with respect to the induced metric.

When the boundary condition is incorporated, we obtain the evolution equation

$$\frac{\partial u}{\partial t} = \triangle_{\mathcal{M}} u + \lambda (N-1)\, H(\cdot) f(u) + \frac{\lambda^2}{2} \frac{d}{du} f^2(u) \quad \text{on} \quad \mathcal{M}.$$

After finding a local minimizer of the functional

$$\mathcal{E}_{\mathcal{M}}(u) \stackrel{\text{def}}{=} \int_{\mathcal{M}} \left\{ \frac{\|\nabla u\|^2}{2\lambda H(\cdot)} - \left[\frac{\lambda f^2(u)}{2H(\cdot)} + (N-1)\, F(u) \right] \right\} dV_g$$

where $\|\bigtriangledown u\|^2 = g^{ik} u_{,i} u_{,k} = u^{,k} u_{,k}$, $u_{,k}$ and $u^{,k}$ are respectively the covariant and contravariant components of the gradient, $F = \int_0 f$, problem (2) is shown to have a pattern, $\overline{u}$ say, for λ large enough and $f(u) = u - u^3$, as long as Ω is:

- strictly convex
- symmetric with respect to a hyperplane through the origin
- $H(\cdot)$ is sufficiently small on two arbitrary disjoint sets separated by this hyperplane

Think of it as an ellipsoid whose vertical axis is much smaller than the other two.

Finally $\overline{u}$ turns out to be the trace of the pattern for (1) we have been looking for.

References

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